

An Exact Inflationary Solution On The Brane

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We study the evolution of universe with a single scalar field of constant potential minimally coupled to gravity in the brane world cosmology. We find an exact inflationary solution which is not in slow roll. We discuss the limiting cases of the solution. We show that at late times the solution is asymptotic to the de Sitter solution independently of the brane tension. For $t \rightarrow 0$ the solution leads to singularity but the nature of the approach to singularity depends upon the brane tension.

I. INTRODUCTION

The higher dimensional cosmological models inspired by string models are under active investigation at present [1,2]. According to the brane World scenario our four dimensional space time is realized as a boundary (brane) of higher dimensional space time (the bulk). The simplest possibility uses the embedding into five dimensional manifold [3]. In these models the Einstein theory of a five dimensional space time is considered with a four dimensional boundary (3 brane) on which the boundary conditions should be treated dynamically. The metric describing the full 4+1 dimensional space is not flat along the extra dimension ; it constitutes a slice of Anti-de Sitter space (AdS). In this picture all the matter fields are confined to the brane while gravity can propagate in the bulk. The small value of the true five dimensional Planck's mass in these models is related to its large effective four dimensional value by the extremely large wrap of five dimensional space . And this offers a possibility for alleviating the hierarchy problem in the particle theory [1]. The field equations on the brane get contribution from dynamics of extra dimension and differ from the usual Einstein equations. This may have important implications for cosmology. For instance , the Friedman equation gets modified by a term quadratic in density which enhances the prospects of inflation [4, 6]. A particular attention has also been paid on the quantum creation of the brane world [5].

The equations of inflation are normally solved in slow roll approximation. In this approximation the slow roll parameters $\epsilon, \eta \ll 1$ [7]. The parameters ϵ and η are related to the slope and the curvature of the field potential respectively. The slow roll conditions in the usual Friedman Cosmology as well as their generalization for the brane world express the necessary but not sufficient conditions for neglecting the acceleration term in comparison of the force and the friction terms in the field equation. It is possible to find a solution of the field equation which satisfies the slow roll conditions but has still a large speed. One such solution in the usual 4-dimensional space time was given in reference [8]. In this paper we obtain a similar solution on the brane.

In the five dimensional brane world scenario the Friedman constraint equation acquires the generalized form ,

$$H^2 = \frac{8\pi}{3M_4^2} \rho (1 + \rho/2\lambda_b) + \frac{\Lambda_4}{3} + \frac{E}{a^4(t)} \quad (1)$$

where E is a constant which describes the bulk graviton effect on a 3-brane and Λ_4 is the four dimensional Planck's constant. The brane tension λ_b relates the four dimensional Planck's mass with its five dimensional counterpart as,

$$M_4 = \sqrt{\frac{3}{4\pi}} (M_5^2 / \sqrt{\lambda_b}) M_5 \quad (2)$$

For simplicity we set Λ_4 equal to zero and also drop the last term as otherwise the inflation would render it so , leading to the expression ,

$$H^2 = \frac{8\pi}{3M_4^2} \rho (1 + \rho/2\lambda_b) \quad (3)$$

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where $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, if one is dealing with a universe dominated by a single scalar field minimally coupled to gravity. The equation of motion of the field propagating on the brane is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \quad (4)$$

II. THE SLOW ROLL EVOLUTION AND THE EQUATION OF STATE ON THE BRANE

The field equation and the Friedman equation on brane in slow roll approximation have the form ,

$$3H\dot{\phi} = -\frac{dV(\phi)}{d\phi} \quad (5)$$

$$H^2 = \frac{8\pi}{3M_4^2}V(1 + V/2\lambda_b) \quad (6)$$

The adiabatic index in slow roll assumes the form ,

$$\gamma = \frac{\dot{\phi}^2/2 - V}{\dot{\phi}^2/2 + V} + 1 \simeq \dot{\phi}^2/V$$

Using Eq. (5) we find

$$\gamma = \left(\frac{2}{3}\right) \frac{M_4^2}{16\pi} \frac{1}{(1 + V/2\lambda_b)} \left(\frac{V_{,\phi}}{V}\right)^2 \quad (7)$$

For inflation to occur in usual four dimensional space time,

$$\epsilon = \frac{3}{2}\gamma_4 < 1 \quad (8)$$

where γ_4 is the adiabatic index in the four dimensional space time and ϵ is the slow roll parameter in the usual case. However, the condition on γ for inflation gets modified on brane. This can be seen from the following. The equation for acceleration on brane takes the form ,

$$\frac{\ddot{a}(t)}{a(t)} = - \left(\frac{8\pi}{6M_4^2} \right) \rho ((1 + 3\omega) + \rho/\lambda_b(2 + 3\omega)) \quad (9)$$

The condition for inflation on brane follows ,

$$\omega < -\frac{1}{3} (1 + 2\rho/\lambda_b) / (1 + \rho/\lambda_b) \quad (10)$$

where

$$\omega = \gamma - 1$$

Or

$$\gamma < \frac{2}{3} (1 + \rho/2\lambda_b) / (1 + \rho/\lambda_b) \quad (11)$$

Similar to usual 4-dimensional case we can again define the slow roll parameter,

$$\epsilon = \frac{3}{2}\gamma_B \quad (12)$$

where,

$$\gamma_B = \gamma \frac{(1 + \rho/\lambda_b)}{(1 + \rho/2\lambda_b)}. \quad (13)$$

γ_B satisfies the same condition for inflation as in usual four dimensional space time and this leads to the correct expression for slow roll parameter [6],

$$\epsilon = \frac{M_4^2}{16\pi} \left(\frac{V_{,\phi}}{V} \right)^2 [(1 + V/\lambda_b)/(1 + V/2\lambda_b)^2] \quad (14)$$

The state equation on brane takes the form ,

$$\omega_B = \gamma_B - 1 \quad (15)$$

and

$$\omega_B \equiv \omega(1 + \rho/\lambda_b)/(1 + \rho/2\lambda_b) \quad (16)$$

which is clear from(13).

III. AN EXACT INFLATIONARY SOLUTION ON THE BRANE

In slow roll approximation one assumes that $V(\phi) \gg \dot{\phi}^2/2$ and the inflaton motion is friction dominated. The acceleration term can then be neglected in equation (4). A necessary but not sufficient condition for this to happen on the brane is given by ,

$$\epsilon = \frac{M_4^2}{16\pi} \left(\frac{V_{,\phi}}{V} \right)^2 \left[\frac{(1 + V/\lambda_b)}{(1 + V/2\lambda_b)^2} \right] < 1 \quad (17)$$

An exact solution of field equations which satisfies (17) but which is not in slow roll i.e $\dot{\phi}$ is not small was written by Faraoni in Friedman cosmology [8]. We find a similar exact solution on the 3-brane. We shall look for an exact solution of evolution equations for a constant potential ,

$$V(\phi) = V_0 = \text{Constant} \quad (18)$$

for $\dot{\phi}$ not equal to zero. In this case the field equation readily solves to give $\dot{\phi}$,

$$\dot{\phi} = \frac{C}{a^3} \quad (19)$$

where C is an integration constant. The Friedman equation with constant potential given by (18) and the expression for the field velocity given by (19) assumes the form ,

$$\frac{dy}{dt} = \beta \sqrt{(\alpha e^{-6y} + 1) (\delta_1 + \delta \alpha e^{-6y})} \quad (20)$$

Where $\alpha = \frac{C^2}{2V_0}$, $y = \ln a$, $\beta = \left(\frac{8\pi V_0^2}{3M_p^2} \right)^{1/2}$, $\delta = \frac{V_0}{2\lambda_b}$ and $\delta_1 = \delta + 1$. Equation(20) can immediately be integrated to yield the expression for the scale factor,

$$a(t) = \left[\left(\frac{\alpha(\delta_1^2 + \delta^2)}{4A\delta_1} - \frac{\delta\alpha}{2A} \right) e^{6\sqrt{\delta_1}\beta t} + \frac{\alpha A}{4\delta_1} e^{-6\sqrt{\delta_1}\beta t} - \frac{\alpha(\delta_1 + \delta)}{2\delta_1} \right]^{1/6} \quad (21)$$

Where the integration constant A is chosen such that a(t) goes to zero for t going to zero. And this fixes the value of A,

$$A = 1 + 2\delta + (2 + 2\delta)\sqrt{\delta/(1 + \delta)} \quad (22)$$

IV. INVESTIGATION OF THE SOLUTION

The solution given by equation (17) has interesting asymptotic forms. For t going to zero the scale factor vanishes and all the physical quantities like pressure and field energy density diverge. The solution crucially depends upon brane parameter δ which contains the brane tension λ_b and the approach to singularity depends upon this parameter. On the other hand at late times $t \rightarrow \infty$ the solution asymptotically approaches the de Sitter solution independently of δ . This is not surprising as it is well known that de Sitter solution is a late time attractor in the 4-dimensional space time and the same seems to be true on the brane also. However, a rigorous phase space analysis analogous to reference [9] is required to be carried out on the brane. It would be interesting to consider the high and low energy limits of the solution (21). In the low energy limit $\delta \ll 1$ i.e $V_0 \ll 2\lambda_b$, the usual Friedman cosmology is retrieved and the expression (21) assumes the form [8],

$$a(t) = \alpha^{1/6} \sinh^{1/3}(3\beta t) \quad (23)$$

The integration constant A in this case turns out to be equal to one. The equation of state in this case is given by,

$$\omega \equiv \frac{p}{\rho} = 1 - 2 \tanh^2(3\beta t) \quad (24)$$

The brane effects are most pronounced in high energy limit $\delta \rightarrow \infty$. In this limit the second term in the expression does not survive. Expanding the coefficient of exponential in the first term into Teller series in $1/\delta$ and keeping the first order term we obtain,

$$a(t) = \alpha^{1/6} \left(e^{6\sqrt{\delta}\beta t} - 1 \right)^{1/6} \quad (25)$$

With $A = 4\delta$. The solution can easily be found out by directly integrating the generalized Friedman equation (3) in high energy limit. The equation of state in this case turns out to be,

$$\frac{p}{\rho} = 2e^{-6\sqrt{\delta}\beta t} - 1 \quad (26)$$

In low energy limit for t going to zero $a(t) \propto t^{1/3}$ where as in high energy limit $a(t) \propto t^{1/6}$ i.e the approach to singularity is slower than the usual big bang scenario. The equation of state similar to Friedman case (Eq(24)) interpolates between the stiff matter equation of state $p = \rho$ and the vacuum state equation $p = -\rho$. The general solution (21) interpolates between the solutions given by (23) and (25) as δ varies between zero (low energy limit) and infinity (high energy limit).

To conclude, we have studied the evolution of the universe with a single scalar field of constant potential on the brane. We have obtained an exact inflationary solution which is asymptotic to the de Sitter solution at late times independently of the brane tension. At $t = 0$, the solution exhibits singularity and the approach to singularity depends upon the brane tension.

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